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A nonlinear equation for fluids in multiconnected domain

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Abstract

In this paper we propose a mathematical model for Reynolds' equation of a compressible fluid on a multiconnected field which simulates the function of a hybrid bearing. The boundary conditions on the inner boundaries are derived from the flow-rate continuity through the supplying orifices and are expressed by means of an integro-differential nonlinear equation. We propose a numerical method to solve this mathematical model.

Keywords: bearing; Reynolds' nonlinear equation; multiconnected field; 'column-wise' technique

1 Introduction

There are many studies dedicated to hybrid bearings. Some of such papers can be found in the complete book [1]. The aim of our present paper is to generalize the studies (done, for instance, in [2–4]) of hybrid bearings. In the previous papers, the domain in which the equation of Reynolds was solved is supposed multiconnected field with inside boundaries represented by continuous lines, so-called 'supplying lines'.

We consider a multiconnected domain in which the inside frontiers have a rectangular form, which has permitted, through passing to the limit of the dimensions and of the number of frontiers, the simulation of a supplying line, so we could compare our results with those of above mentioned researchers. The physical situation considered in this paper has the feature that the boundary conditions on the inner boundaries are of more difficult forms. These boundary conditions become nonlinear integral-differential expressions with respect to the solution of the Reynolds equation. With the aim to compare our results with those reported in the open literature, we consider that the inner boundaries are disposed in a normal plane to the bearing surfaces, but it is worth to mention that this is not a mandatory condition for our method. An iterative procedure of linearization, as proposed in [5–7], is used in the present study. The resulting algebraic system is solved on using a 'column-wise' technique (see, for instance, Castelli and Pirvies [2]).

In [8] the authors use the Cesaro means for the components of energy and prove the asymptotic equipartition in mean of the kinetic and strain energies. Some results from the theory of semigroups of linear operators are used in [9] in order to prove the existence and uniqueness of a weak solution. The paper [10] is dedicated to heat conduction in deformable bodies which depends on two temperatures. The first one is the conductive

temperature, the second is the thermodynamic temperature, and the difference between them is proportional to the heat supply.

Also, we must outline the minimum principle in [11] and remark that the solution proposed in [12] can be used for Maxwell's fluids and, also, for Newtonian fluids.

2 Basic notions

Consider a circular bearing supplying with a compressible fluid through feed pocket disposed in planes normal on the bearing axis. The geometrical elements which appear in our mathematical model, are:

- L = the length of the bearing;
- R = the radius of the shaft;
- C = the radial clearance;
- ε = the eccentricity;
- c = the length of a feed pocket;
- d = the width of a feed pocket;
- θ = the polar angle measured from the line of centers bearing surfaces;
- H = the thickness of fluid film, $H = C + \varepsilon \cos \theta$.

We use a Cartesian system of orthogonal axes such that the Ox_1 and Ox_2 are included in the bearing shaft surface, with the origin O in the intersection of the normal plane on the rotation axis, in an end of the bearing, with the plane determined by the axes of bearing shaft and bearing baking. Also, the Ox_3 -axis is parallel with the rotation axis, the Ox_1 has the sense of the rotation of bearing shaft. In this system the coordinates of the centers of the feed pockets are (x_{1m}, x_{3m}) , where m is a generic index. If the number of feed pockets is n_0 , then $m = \overline{1, n_0}$. The domain represented by the surface of the bearing shaft, on which are projected the feed pockets can be assimilated with a domain of band type, having width L in the direction of Ox_3 and being periodic in the direction of Ox_1 , with period $2\pi R$. Based on the periodicity, we can consider the mathematical model in the domain

$$\mathcal{D} = [0, 2\pi] \times [0, L] \setminus \bigcup_m D_m,$$

where

$$D_m = \left(x_{1m} - \frac{c}{2}, x_{1m} + \frac{c}{2}\right) \times \left(x_{3m} - \frac{d}{2}, x_{3m} + \frac{d}{2}\right),$$

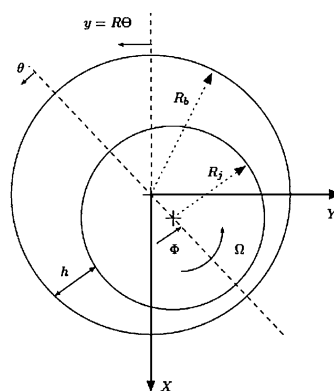
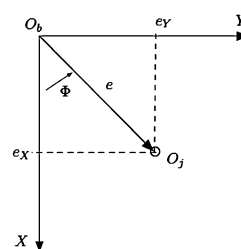
with c the length of a feed pocket and d the width of a feed pocket.

3 The mathematical model

We use the notations:

- P = the pressure in the fluid film;
- H = the gap between the rotating shaft and the bushing;
- ϱ = the density of the fluid;
- μ = the dynamic viscosity of the fluid;
- V = the relative speed of the axis bearing;
- p_a = the pressure on the ends of the bearing.

With these notations, the mathematical model becomes [11]:

Figure 1 Journal bearing schematic.**Figure 2 Definition of the journal eccentricity.**

- the Reynolds equation for the steady state condition of a gas journal bearing:

$$\frac{\partial}{\partial x_1} \left(PH^3 \frac{\partial P}{\partial x_1} \right) + \frac{\partial}{\partial x_3} \left(PH^3 \frac{\partial P}{\partial x_3} \right) = 6\mu V \frac{\partial}{\partial x_1} (PH); \quad (1)$$

- the equation of state, which is supplemented with the assumption that the gas undergoes an isothermal process:

$$\varrho = \frac{P}{RT}; \quad (2)$$

- the conditions on boundary of the domain \mathcal{D} , where (1) is applied:

$$P(x_1, 0) = P(x_1, L) = p_a; \quad (3)$$

$$\oint_{C_m} \varrho \left(\frac{\mathbf{V}H}{2} - \frac{H^3}{12\mu} \text{grad } P \right) \mathbf{n} dl = q_m, \quad (x_1, x_3) \in C_m, \quad (4)$$

where the integral represents the flow-rate through the surface into the surface determined by C_m , the boundary of the domain D_m , and the generators of length H . q_m is the flow-rate through the incoming orifice (\mathbf{n} is the outside normal of the curve C_m).

The location of the minimum film thickness is defined by the attitude angle Φ (see Figures 1 and 2). The film thickness function $h = h(x_3, x_1)$ depends on the position of the center of the shaft according to

$$h = C + e_x \cos(\Theta) + e_y \sin(\Theta),$$

where $C = R_b - R_j$ is the radial clearance of the bearing and $\Theta = \theta + \Phi$.

In the equations of the mathematical model, we use the following non-dimensional variables:

$$\theta = \frac{x_1}{R}, \quad h = \frac{H}{C} = 1 + e \cos \theta, \quad \omega = \frac{V}{R}, \quad z = \frac{x_3}{R}, \quad p = \frac{P}{p_a}.$$

With these variables, (1) and the conditions (2) and (3) become

$$\frac{\partial}{\partial \theta} \left(p h^3 \frac{\partial p}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(p h^3 \frac{\partial p}{\partial z} \right) = \Lambda \frac{\partial}{\partial \theta} (p h), \quad (5)$$

$$p(\theta, 0) = p\left(\theta, \frac{L}{R}\right) = 1, \quad (6)$$

$$\oint_{C_m} (\Lambda p h \mathbf{i}_1 - R h^3 p \operatorname{grad} P) \mathbf{n} d\mathbf{l} = \mathbf{q}_m, \quad (7)$$

where

$$\begin{aligned} \Lambda &= \frac{6\mu\omega}{p_a} \left(\frac{R}{C}\right)^2 \text{ is the bearing number,} \\ q_m &= \frac{2\pi \Lambda_s}{n_0} \left(\frac{p_s}{p_a}\right)^2 \sqrt{\frac{1+\delta^2}{h_m^2 + \delta^2}} h_m G_m, \\ \delta &= \frac{a^2}{dC}, \\ \Lambda_s &= \frac{6\mu n_0 a^2}{p_s C^3} \sqrt{\frac{RT}{1+\delta^2}}, \\ G_m &= C_D \gamma^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}, \quad \text{if } \frac{p_m}{p_s} \leq \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}, \\ G_m &= C_D \left\{ \frac{2\gamma}{\gamma-1} \left(\frac{p_m}{p_s}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p_m}{p_s}\right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}}, \quad \text{if } \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} < \frac{p_m}{p_s} \leq 1; \end{aligned} \quad (8)$$

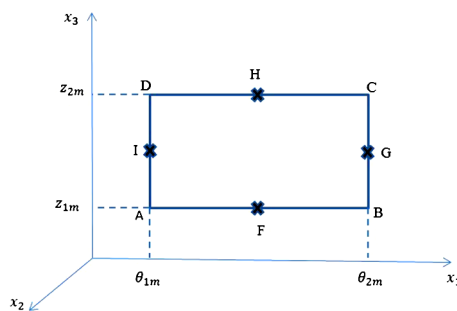
where θ and z are the coordinates in the circumferential and axial direction, respectively; c is the radial clearance, h is the thickness of the fluid film on the ridge region;

- p_s = the supplying pressure;
- p_m = the pressure on the curve C_m ;
- h_m = the film thickness on the curve C_m ;
- a = the radius of the feed orifice;
- γ = the adiabatic constant and its value is about 1,405;
- C_D = a coefficient, depending on the form of the feed orifice, experimentally was obtained: $C_D \in [0.6, 1]$.

Because of the fact that the coefficients of (5) depend only on the parameter Λ , which is constant with regard to θ and z , and with regard to h , which is a periodical function on θ , one finds, as a result, the solution of the equation is periodical:

$$p(\theta, z) = p(\theta + 2\pi, z). \quad (9)$$

It is easy to see that the equations of the mathematical model are nonlinear. The linearization of these equations is made, in the usual way, by introducing the function $Q = (ph)^2$.

Figure 3 The representation of the curve C_m .

With this new variable, (5), (6), and (9) become

$$\frac{\partial^2 Q}{\partial \theta^2} + \frac{\partial^2 Q}{\partial z^2} - \frac{1}{h} \left(\frac{\partial h}{\partial \theta} + \frac{\Lambda}{ph} \right) \frac{\partial Q}{\partial \theta} - \frac{2}{h} \frac{\partial^2 h}{\partial \theta^2} Q = 0, \quad (10)$$

$$Q(\theta, 0) = Q\left(\theta, \frac{L}{R}\right) = h^2, \quad (11)$$

$$Q(\theta, z) = Q(\theta + 2\pi, z). \quad (12)$$

In order to obtain the new form of (7), we consider the domain corresponding to a feed pocket.

Suppose that the pressure is constant on the boundary of the feed pocket,

$$p(\theta, z) = p_m, \quad (\theta, z) \in C_m,$$

and consider a mean value of h on C_m , that is, \tilde{h}_m , and a mean value of q_m on C_m , that is, \tilde{q}_m . Then we have as a result a constant value for Q :

$$Q(\theta, z) = (p_m \tilde{h}_m)^2 = Q_m, \quad (\theta, z) \in C_m.$$

The outline C_m is a rectangle with the sides $AB \parallel CD \parallel Ox_1$ and $BC \parallel DA \parallel Ox_3$ (see Figure 3).

The equations of the sides are:

- AB : $z = z_{1m}$, $\theta \in [\theta_{1m}, \theta_{2m}]$;
- BC : $\theta = \theta_{2m}$, $z \in [z_{1m}, z_{2m}]$;
- CD : $z = z_{2m}$, $\theta \in [\theta_{1m}, \theta_{2m}]$;
- DA : $\theta = \theta_{1m}$, $z \in [z_{1m}, z_{2m}]$.

With these hypotheses, (7) becomes

$$\begin{aligned} \mathbf{q}_m &= \oint_{C_m} (\Lambda p h \mathbf{i}_1 - R h^3 p \operatorname{grad} P) \mathbf{n} d\mathbf{l} \\ &= \int_{AB} \left[\Lambda p h \mathbf{i}_1 - h^3 p \left(\frac{\partial p}{\partial \theta} \mathbf{i}_1 + \frac{\partial p}{\partial z} \mathbf{i}_3 \right) \right] \mathbf{n}_{AB} d\mathbf{l} \\ &\quad + \int_{BC} \left[\Lambda p h \mathbf{i}_1 - h^3 p \left(\frac{\partial p}{\partial \theta} \mathbf{i}_1 + \frac{\partial p}{\partial z} \mathbf{i}_3 \right) \right] \mathbf{n}_{BC} d\mathbf{l} \\ &\quad + \int_{CD} \left[\Lambda p h \mathbf{i}_1 - h^3 p \left(\frac{\partial p}{\partial \theta} \mathbf{i}_1 + \frac{\partial p}{\partial z} \mathbf{i}_3 \right) \right] \mathbf{n}_{CD} d\mathbf{l} \end{aligned}$$

$$\begin{aligned}
& + \int_{DA} \left[\Lambda p h \mathbf{i}_1 - h^3 p \left(\frac{\partial p}{\partial \theta} \mathbf{i}_1 + \frac{\partial p}{\partial z} \mathbf{i}_3 \right) \right] \mathbf{n}_{DA} d\mathbf{l} \\
& = \int_{AB} h^3 p \frac{\partial p}{\partial z} d\theta + \int_{BC} \left(\Lambda p h - h^3 p \frac{\partial p}{\partial \theta} \right) dz \\
& \quad + \int_{CD} h^3 p \frac{\partial p}{\partial z} d\theta + \int_{DA} \left(\Lambda p h - h^3 p \frac{\partial p}{\partial \theta} \right) dz, \\
\tilde{q}_m & = \frac{1}{2h_m^2} \left[\int_{AB} h^3 \frac{\partial Q}{\partial z} d\theta + \frac{2\Lambda h_G}{p_m} (z_C - z_B) Q_m - h_G^3 \int_{BC} \frac{\partial Q}{\partial \theta} dz \right. \\
& \quad \left. + \int_{CD} h^3 \frac{\partial Q}{\partial \theta} d\theta + \frac{2\Lambda h_1}{p_m} (z_A - z_D) Q_m - h_1^3 \int_{DA} \frac{\partial Q}{\partial \theta} dz \right].
\end{aligned}$$

We approximate the derivatives of the function Q on the outline by the values in the points F, G, H, I (the mid-points of the sides AB, BC, CD, DA):

$$\begin{aligned}
\left(\frac{\partial Q}{\partial z} \right)_{AB} & \approx \left(\frac{\partial Q}{\partial z} \right)_F, & \left(\frac{\partial Q}{\partial \theta} \right)_{BC} & \approx \left(\frac{\partial Q}{\partial \theta} \right)_G, \\
\left(\frac{\partial Q}{\partial z} \right)_{CD} & \approx \left(\frac{\partial Q}{\partial z} \right)_H, & \left(\frac{\partial Q}{\partial \theta} \right)_{DA} & \approx \left(\frac{\partial Q}{\partial \theta} \right)_I,
\end{aligned}$$

such that, for the flow, we obtain the expression

$$\begin{aligned}
\tilde{q}_m & = \frac{1}{2h_m^2} \left\{ \left[\left(\frac{\partial Q}{\partial z} \right)_F - \left(\frac{\partial Q}{\partial z} \right)_H \right] \int_{AB} h^3 d\theta \right. \\
& \quad \left. - \bar{BC} \left[h_G^3 \left(\frac{\partial Q}{\partial \theta} \right)_G - h_1 \left(\frac{\partial Q}{\partial \theta} \right)_I \right] + \frac{2\Lambda \bar{BC}}{p_m} (h_G - h_1) Q_m \right\}. \quad (13)
\end{aligned}$$

To obtain the convergence of our procedure, the function G_m from (8) is prolonged on the domain $p_m/p_s > 1$.

4 The solution

In order to solve the system of equations of the mathematical model, we use the method of finite difference. The domain \mathcal{D} is covered with a grid of parallel lines with Ox_1 and Ox_3 . The feed pocket center is a node. We have

$$\begin{aligned}
G_m & = C_D \gamma^{1/2} \left(\frac{2}{\gamma+1} \right)^{(\gamma+1)/2(\gamma-1)}, & \text{if } \frac{p_m}{p_s} \leq \left(\frac{2}{\gamma+1} \right)^{\gamma/(\gamma-1)}, \\
G_m & = C_D \left\{ \frac{2\gamma}{\gamma-1} \left(\frac{p_m}{p_s} \right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p_m}{p_s} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}}, & \text{if } \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}} < \frac{p_m}{p_s} \leq 1, \\
G_m & = -C_D \left\{ \frac{2\gamma}{\gamma-1} \left(\frac{p_m}{p_s} \right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{p_m}{p_s} \right)^{\frac{\gamma-1}{\gamma}} \right] \right\}^{\frac{1}{2}}, & \text{if } \frac{p_m}{p_s} > 1.
\end{aligned} \quad (14)$$

If the dimensions of the grid are $N \times M$, then the coordinates of a node are

$$\begin{aligned}
\theta_j & = (j-1)\Delta\theta, & j & = \overline{1, N}, \\
z_i & = (i-1)\Delta z, & i & = \overline{1, M},
\end{aligned}$$

where

$$\Delta\theta = \frac{2\pi}{N-1}, \quad \Delta z = \frac{L}{R(M-1)}.$$

If we use the notations $Q(\theta_j, z_i) = Q_{j,i}$, then the derivatives can be written as follows:

$$\begin{aligned} \left(\frac{\partial Q}{\partial \theta}\right)_{j,i} &= \frac{Q_{j+1,i} - Q_{j-1,i}}{2\Delta\theta}, & \left(\frac{\partial Q}{\partial z}\right)_{j,i} &= \frac{Q_{j,i+1} - Q_{j,i-1}}{2\Delta z}, \\ \left(\frac{\partial^2 Q}{\partial \theta^2}\right)_{j,i} &= \frac{Q_{j+1,i} - 2Q_{j,i} + Q_{j-1,i}}{(\Delta\theta)^2}, & \left(\frac{\partial^2 Q}{\partial z^2}\right)_{j,i} &= \frac{Q_{j,i+1} - 2Q_{j,i} + Q_{j,i-1}}{(\Delta z)^2}. \end{aligned} \quad (15)$$

In the neighborhood of the feed pocket, the formulas will be modified.

The derivatives of the functions Q , in the node (k, l) in the neighborhood of the outline C_m can be calculated as follows:

- if $(l-1) \cdot \Delta z \in [z_{1m}, z_{2m}]$ and $0 < k \cdot \Delta\theta - \theta_{m1} = \Delta\theta_1 < \Delta\theta$, then

$$\begin{aligned} \left(\frac{\partial Q}{\partial \theta}\right)_{k,l} &= \frac{\Delta\theta}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_m - \frac{\Delta\theta - \Delta\theta_1}{\Delta\theta \Delta\theta_1} Q_{k,l} - \frac{\Delta\theta_1}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_{k-1,l}, \\ \left(\frac{\partial^2 Q}{\partial \theta^2}\right)_{k,l} &= \frac{2}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_m - \frac{2}{\Delta\theta \Delta\theta_1} Q_{k,l} - \frac{2}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_{k-1,l}; \end{aligned} \quad (16)$$

- if $(l-1) \cdot \Delta z \in [z_{1m}, z_{2m}]$ and $0 < (k-1) \cdot \Delta\theta - \theta_{m2} < \Delta\theta$, then

$$\begin{aligned} \left(\frac{\partial Q}{\partial \theta}\right)_{k,l} &= \frac{\Delta\theta_1}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_{k+1,l} + \frac{\Delta\theta - \Delta\theta_1}{\Delta\theta \Delta\theta_1} Q_{k,l} - \frac{\Delta\theta}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_m, \\ \left(\frac{\partial^2 Q}{\partial \theta^2}\right)_{k,l} &= \frac{2}{\Delta\theta(\Delta\theta + \Delta\theta_1)} Q_{k+1,l} - \frac{2}{\Delta\theta \Delta\theta_1} Q_{k,l} + \frac{2}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} Q_m. \end{aligned} \quad (17)$$

With a similar procedure we can obtain the derivatives with regard to z :

- if $(k-1) \cdot \Delta\theta \in [\theta_{m1}, \theta_{m2}]$ and $0 < l \cdot \Delta z - z_{m1} = \Delta z_1 < \Delta z$, then

$$\left(\frac{\partial Q}{\partial z}\right)_{k,l} = \frac{\Delta z}{\Delta z_1(\Delta z + \Delta z_1)} Q_m - \frac{\Delta z - \Delta z_1}{\Delta z \Delta z_1} Q_{k,l} - \frac{\Delta z_1}{\Delta z_1(\Delta z + \Delta z_1)} Q_{k,l-1}; \quad (18)$$

- if $(k-1) \cdot \Delta\theta \in [\theta_{m1}, \theta_{m2}]$ and $0 < (l-1) \cdot \Delta z - z_{m2} < \Delta z$, then

$$\begin{aligned} \left(\frac{\partial Q}{\partial z}\right)_{k,l} &= \frac{\Delta z_1}{\Delta z_1(\Delta z + \Delta z_1)} Q_{k+1,l} + \frac{\Delta z - \Delta z_1}{\Delta z \Delta z_1} Q_{k,l} - \frac{\Delta z}{\Delta z_1(\Delta z + \Delta z_1)} Q_m, \\ \left(\frac{\partial^2 Q}{\partial z^2}\right)_{k,l} &= \frac{2}{\Delta z(\Delta z + \Delta z_1)} Q_{k+1,l} - \frac{2}{\Delta z \Delta z_1} Q_{k,l} + \frac{2}{\Delta z_1(\Delta z + \Delta z_1)} Q_m. \end{aligned} \quad (19)$$

In order to determine Q_m , we made in (13) the approximations

$$\begin{aligned} \left(\frac{\partial Q}{\partial z}\right)_F &= \frac{Q_m - Q_{j,i1}}{\Delta z_1}, & \left(\frac{\partial Q}{\partial z}\right)_H &= \frac{Q_{j,i2} - Q_m}{\Delta z_1}, \\ \left(\frac{\partial Q}{\partial \theta}\right)_I &= \frac{Q_m - Q_{j,i}}{\Delta\theta_1}, & \left(\frac{\partial Q}{\partial \theta}\right)_G &= \frac{Q_{j2,i} - Q_m}{\Delta\theta_1}, \end{aligned} \quad (20)$$

where

$$i_1 = \frac{z_{m_1} - \Delta z_1}{\Delta z} + 1, \quad i_2 = \frac{z_{m_2} + \Delta z_1}{\Delta z} + 1,$$

$$j_1 = \frac{\theta_{m_1} - \Delta \theta_1}{\Delta \theta} + 1, \quad j_2 = \frac{\theta_{m_2} + \Delta \theta_1}{\Delta \theta} + 1.$$

Substituting h_m with \tilde{h}_m in the expression of \tilde{q}_m from (13), one finds, as a result,

$$\begin{aligned} & \frac{4\pi \Lambda_s p_s \tilde{h}_m^3}{n_0} \sqrt{\frac{1 + \delta^2}{\tilde{h}_m^3 + \delta^2}} G_m \\ &= \left[\frac{2}{\Delta z_1} \int_{\theta_{m_1}}^{\theta_{m_2}} h^3 d\theta + \frac{z_{m_2} - z_{m_1}}{\Delta \theta_1} (h_G^3 + h_1^3) + \frac{2\Lambda(z_{m_2} - z_{m_1})}{p_m} (h_G - h_1) \right] Q_m \\ & \quad - \frac{1}{zP} \int_{\theta_{m_1}}^{\theta_{m_2}} h^3 d\theta (Q_{j,i_1} + Q_{j,i_2}) - \frac{z_{m_2} - z_{m_1}}{\Delta \theta_1} (h_G^3 Q_{j_2,i} + h_1^3 Q_{j_1,i}). \end{aligned} \quad (21)$$

As an initial iteration we use the function $Q^{(0)}$ which satisfies the boundary conditions (11) and the periodicity conditions (12), such that $Q^{(0)} = h^2$. In (21) we consider the unknown variable $x = p_m/p_s$. The coefficients of this variable can be calculated as a function of the previous iteration, such that (21) can be written in the form

$$AG_m(x) - B^{(r-1)}x^2 + C^{(r-1)} = 0, \quad (22)$$

where

$$\begin{aligned} A &= \frac{4\pi \Lambda_s p_s^2 \tilde{h}_m^3}{n_0} \sqrt{\frac{1 + \delta^2}{\tilde{h}_m^3 + \delta^2}}, \\ B &= \left[\frac{2}{\Delta z_1} \int_{\theta_{m_1}}^{\theta_{m_2}} h^3 d\theta + \frac{z_{m_2} - z_{m_1}}{\Delta \theta_1} (h_G^3 + h_1^3) \right. \\ & \quad \left. + \frac{2\Lambda(z_{m_2} - z_{m_1})}{p_m} (h_G - h_1) \right] p_s^2 \tilde{h}_m^2, \\ C &= \frac{1}{\Delta z_1} \int_{\theta_{m_1}}^{\theta_{m_2}} h^3 d\theta (Q_{j,i_1}^{(r-1)} + Q_{j,i_2}^{(r-1)}) \\ & \quad + \frac{z_{m_2} - z_{m_1}}{\Delta \theta_1} (h_G^3 Q_{j_2,i}^{(r-1)} + h_1^3 Q_{j_1,i}^{(r-1)}), \quad r \geq 1. \end{aligned} \quad (23)$$

The solution of (22) is obtained by using the chord method, because $x \in [0, 1]$. In the case $x > 1$, by using the transformation $x = 1/x'$ and (22),

$$Ax'^2 G(x') - Cx'^2 + B = 0. \quad (24)$$

After (22) or (24) is solved, we can calculate Q_m :

$$Q_m = \begin{cases} (p_s x \tilde{h}_m)^2, & \text{if } x \leq 1, \\ (p_s (x')^{-1} \tilde{h}'_m)^2, & \text{if } x > 1. \end{cases} \quad (25)$$

In the inside points of the domain \mathcal{D} , the algebraic system of equations, which results by using the finite difference in (10), is

$$\begin{aligned} & \frac{1}{\Delta z^2} Q_{j,i-1}^{(r)} - 2 \left[\frac{1}{\Delta \theta^2} + \frac{1}{\Delta z^2} + \frac{1}{h_{j,i}} \left(\frac{\partial^2 h}{\partial \theta^2} \right)_{j,i} \right] Q_{j,i}^{(r)} + \frac{1}{\Delta z^2} Q_{j,i+1}^{(r)} \\ & + \left\{ \frac{1}{\Delta \theta^2} + \frac{1}{\Delta \theta h_{j,i}} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \right\} Q_{j-1,i}^{(r)} \\ & + \left\{ \frac{1}{\Delta \theta^2} - \frac{1}{\Delta \theta h_{j,i}} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \right\} Q_{j+1,i}^{(r)} \\ & = 0, \end{aligned} \quad (26)$$

where $j = \overline{1, N}$, $i = \overline{1, M-1}$.

For $j = 1$ and $j = N$ in (26) is used the periodicity condition:

$$Q_{0,i-1}^{(r)} = Q_{N,i-1}^{(r)}, \quad Q_{N+1,i}^{(r)} = Q_{1,i}^{(r)}. \quad (27)$$

For $i = 1$ and $i = M$ the conditions on the outside boundary become

$$Q_{j,1}^{(r)} = h_j^2, \quad Q_{j,M}^{(r)} = h_j^2. \quad (28)$$

In the neighborhood of C_m the equations are:

- if $(i-1) \cdot \Delta z \in [z_{m1}, z_{m2}]$ and $0 < j \cdot \Delta \theta - \theta_{m2} < \Delta \theta$, then

$$\begin{aligned} & \frac{1}{\Delta z^2} Q_{j,i-1}^{(r)} - 2 \left\{ \frac{1}{\Delta \theta \Delta \theta_1} + \frac{1}{\Delta z^2} + \frac{1}{2h_{j,i}} \frac{\Delta \theta - \Delta \theta_1}{\Delta \theta \Delta \theta_1} \right. \\ & \quad \times \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] + \frac{1}{h_{j,i}} \left(\frac{\partial^2 h}{\partial \theta^2} \right)_{j,i} \left. \right\} Q_{j,i}^{(r)} \\ & + \frac{1}{\Delta z^2} Q_{j,i+1}^{(r)} + \left\{ \frac{2}{\Delta \theta (\Delta \theta + \Delta \theta_1)} + \frac{1}{h_{j,i}} \right. \\ & \quad \times \frac{\Delta \theta_1}{\Delta \theta (\Delta \theta + \Delta \theta_1)} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \left. \right\} Q_{j-1,i}^{(r)} \\ & = \left\{ -\frac{2}{\Delta \theta (\Delta \theta + \Delta \theta_1)} + \frac{1}{h_{j,i}} \frac{\Delta \theta}{\Delta \theta_1 (\Delta \theta + \Delta \theta_1)} \right. \\ & \quad \times \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \left. \right\} Q_m^{(r)}; \end{aligned} \quad (29)$$

- if $(i-1) \cdot \Delta z \in [z_{m1}, z_{m2}]$ and $0 < (j-1) \cdot \Delta \theta - \theta_{m2} < \Delta \theta$, then

$$\begin{aligned} & \frac{1}{\Delta z^2} Q_{j,i-1}^{(r)} - 2 \left\{ \frac{1}{\Delta \theta \Delta \theta_1} + \frac{1}{\Delta z^2} + \frac{1}{2h_{j,i}} \frac{\Delta \theta - \Delta \theta_1}{\Delta \theta \Delta \theta_1} \right. \\ & \quad \times \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] + \frac{1}{h_{j,i}} \left(\frac{\partial^2 h}{\partial \theta^2} \right)_{j,i} \left. \right\} Q_{j,i}^{(r)} \\ & + \frac{1}{\Delta z^2} Q_{j,i+1}^{(r)} + \left\{ \frac{2}{\Delta \theta (\Delta \theta + \Delta \theta_1)} - \frac{1}{h_{j,i}} \right. \end{aligned}$$

$$\begin{aligned} & \times \frac{\Delta\theta_1}{\Delta\theta(\Delta\theta + \Delta\theta_1)} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \Big\} Q_{j+1,i}^{(r)} \\ & = \left\{ -\frac{2}{\Delta\theta(\Delta\theta + \Delta\theta_1)} - \frac{1}{h_{j,i}} \frac{\Delta\theta}{\Delta\theta_1(\Delta\theta + \Delta\theta_1)} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \right\} Q_m^{(r)}; \end{aligned} \quad (30)$$

- if $(j-1) \cdot \Delta\theta \in [\theta_{m1}, \theta_{m2}]$ and $0 < i \cdot \Delta z - z_{m2} < \Delta z$, then

$$\begin{aligned} & \frac{2}{\Delta z(\Delta z + \Delta z_1)} Q_{j,i+1}^{(r)} - 2 \left[\frac{1}{\Delta\theta^2} + \frac{1}{\Delta z \Delta z_1} + \frac{1}{h_{j,i}} \left(\frac{\partial^2 h}{\partial \theta^2} \right)_{j,i} \right] Q_{j,i}^{(r)} \\ & + \left\{ \frac{1}{\Delta\theta^2} + \frac{1}{2\Delta\theta h_{j,i}} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \right\} Q_{j-1,i}^{(r)} \\ & + \left\{ \frac{1}{\Delta\theta^2} - \frac{1}{2\Delta\theta h_{j,i}} \left[\left(\frac{\partial h}{\partial \theta} \right)_{j,i} + \frac{\Lambda}{(Q_{j,i}^{(r-1)})^{1/2}} \right] \right\} Q_{j+1,i}^{(r)} \\ & = -\frac{2}{\Delta z_1(\Delta z + \Delta z_1)} Q_m^{(r)}. \end{aligned} \quad (31)$$

The system of equations (26)-(31) can be written in a matrix form:

$$[A_j]\{Q_j\} + [B_j]\{Q_{j-1}\} + [C_j]\{Q_{j+1}\} = \{R_j\}, \quad (32)$$

where $[A_j]$, $[B_j]$, $[C_j]$ are square matrices of dimension $M \times M$ and $\{Q_j\}$, $\{Q_{j-1}\}$, $\{Q_{j+1}\}$, $\{R_j\}$ are the column vectors of the unknown variable on the column j , $j-1$, $j+1$, and, respectively, the column vector of terms in the right-sides. Here the subscripts represent the iteration order. In (32) and in the relations that follow, we consider the iteration of order r .

In order to solve the system of equations (32) we use the method called 'column-wise'. This method is based on the relation which can be written for each j :

$$\{Q_{j-1}\} = [E_j]\{Q_j\} + \{F_j\} + [D_j]\{Q_N\}. \quad (33)$$

Substituting (33) in (32), one finds, as a result,

$$([A_j] + [B_j][E_j])\{Q_j\} = -[C_j]\{Q_{j+1}\} + (\{R_j\} - [B_j]\{F_j\}) - [B_j][D_j]\{Q_N\}. \quad (34)$$

By solving with regard to $\{Q_j\}$ and comparing with (33) we obtain the relations

$$[E_{j+1}] = -[T_j][C_j], \quad (35)$$

$$[F_{j+1}] = [T_j](\{R_j\} - [B_j]\{F_j\}), \quad (36)$$

$$[D_{j+1}] = -[T_j][B_j][D_j], \quad (37)$$

where

$$[T_j] = ([A_j] + [B_j][E_j])^{-1}. \quad (38)$$

The initial values for (35)-(38) are

$$[E_1] = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \{F_1\} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad [D_1] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (39)$$

After the calculation of the matrix $[E_j]$, $[D_j]$, $\{F_j\}$, for $j = \overline{1, N}$, we write the relation for $j = N + 1$ and use the periodicity condition $\{Q_{N+1}\} = \{Q_1\}$. We find, as a result,

$$\{Q_N\} = (I - [D_{N+1}])^{-1}([E_{N+1}]\{Q_1\} + \{F_{N+1}\}). \quad (40)$$

From (33) and (40), written for $j = N, N-1, \dots, 2$, we obtain the following relation between $\{Q_j\}$ and $\{Q_1\}$:

$$\{Q_j\} = [G_j]\{Q_1\} + \{S_j\}, \quad (41)$$

where

$$[G_N] = (I - [D_{N+1}])^{-1}[E_{N+1}], \quad (42)$$

$$\{S_N\} = (I - [D_{N+1}])^{-1}\{F_{N+1}\}. \quad (43)$$

By means of (33) and (41) we obtain the relation for the calculation of $[G_j]$ and $\{S_j\}$, for $j = N, N-1, \dots, 2$:

$$[G_{j-1}] = [E_j][G_j] + [D_j][G_N], \quad (44)$$

$$\{S_{j-1}\} = [E_j]\{S_j\} + \{F_j\} + [D_j]\{S_N\}. \quad (45)$$

After we compute the matrix $[G_j]$ and $\{S_j\}$, from (41), written for $j = 1$, one finds, as a result,

$$\{Q_1\} = (I - [G_1])^{-1}\{S_1\}. \quad (46)$$

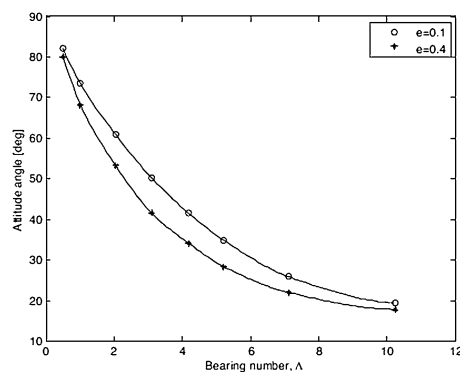
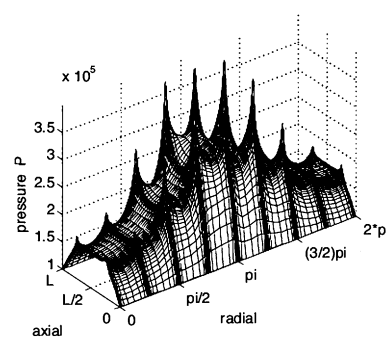
The other vectors $\{Q_j\}$ can be computed now from (41) for $j = \overline{2, N}$. The solution $Q^{(r)}$ is compared with $Q^{(r-1)}$. Our problem is solved if

$$\frac{\sum_{j,i} |Q_{j,i}^{(r)} - Q_{j,i}^{(r-1)}|}{\sum_{j,i} |Q_{j,i}^{(r-1)}|} < \varepsilon, \quad (47)$$

where ε is a positive number of a suitable choice.

If the last condition (47) is not valid, then the calculation is repeated for a new iteration $r + 1$.

Figure 4 compares the attitude angle function of the bearing number for a compliant journal bearing at different eccentricity: 0.1, 0.4. Figure 5 represents the three dimensional plot of the hydrodynamic pressure distribution for the compliant journal bearing

Figure 4 Numerical calculated attitude angles.**Figure 5** Numerical calculated pressure profiles.

running at 30,000 rpm. The hydrodynamic pressure developed along the axial direction has a parabolic profile wherein the pressure is maximum at the mid-section or at the center of the bearing and the pressure is normalized at the ends of the bearing.

5 Conclusions

Even the boundary conditions become nonlinear integral-differential expressions with respect to the Reynolds equation, we can solve the problem by using the ‘column-wise’ technique.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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